

Reals n -generic relative to some perfect tree

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Abstract

We say that a real X is n -generic relative to a perfect tree T if X is a path through T and for all $\Sigma_n^0(T)$ sets S , there exists a number k such that either $X|k \in S$ or for all $\sigma \in T$ extending $X|k$ we have $\sigma \notin S$. A real X is n -generic relative to some perfect tree if there exists such a T . We first show that for every number n all but countably many reals are n -generic relative to some perfect tree. Second, we show that proving this statement requires $\text{ZFC}^- + \text{“}\exists \text{ infinitely many iterates of the power set of } \omega\text{”}$. Third, we prove that every finite iterate of the hyperjump, $\mathcal{O}^{(n)}$, is not 2-generic relative to any perfect tree and for every ordinal α below the least λ such that $\sup_{\beta < \lambda} (\beta \text{th admissible}) = \lambda$, the iterated hyperjump $\mathcal{O}^{(\alpha)}$ is not 5-generic relative to any perfect tree. Finally, we demonstrate some necessary conditions for reals to be 1-generic relative to some perfect tree.

1 Introduction

A real (viewed as an element of 2^ω) is n -generic if for every Σ_n^0 set there is an initial segment of the real which either meets the set or for which no extension of the segment can meet the set. These reals have many interesting characteristics and have been studied extensively (see Jockusch and Posner [3] and Kumabe [5] among others). While the set of n -generics is comeager, it is in some ways limited. In particular, it is completely excluded from the cone above $\mathbf{0}'$ since no 1-generic can compute an r.e. set.

A question which naturally arises then is how this set might be expanded from reals which are n -generic to those that can be made to seem n -generic in some appropriate context. An attractive framework for this question is to consider reals which are n -generic when viewed as paths through a given perfect tree, rather than all of 2^ω .

Definition: A real X is n -generic relative to a perfect tree T if X is a path through T and for all $\Sigma_n^0(T)$ sets S , there is a k such that either $X|k \in S$ or $\sigma \notin S$ for every $\sigma \in T$ extending $X|k$.

Definition: A real X is n -generic relative to some perfect tree if there exists a perfect tree T such that X is n -generic relative to T .

This definition results in a version of genericity which includes many reals which seem “essentially” generic. For example, if G is a n -generic real and R is recursive then neither $G \oplus R$ nor $G \oplus G$ is n -generic, but both are n -generic relative to some perfect tree. Another form of relative genericity that has been studied is genericity relative to another real. Given a real A we say that a real X is n -generic(A) if for every $\Sigma_n^0(A)$ set S , either X meets S or there is an initial segment of X which has no extensions in S . This results in a more restrictive version of genericity which is similar to higher levels of genericity. For example a real is 1-generic($0'$) if and only if it is 2-generic. We note that every real which is n -generic(A) for some real A is n -generic and every real which is n -generic is n -generic relative to some perfect tree.

In this paper, we examine the set of reals n -generic relative to some perfect tree. We first show that the set of reals not n -generic relative to any perfect tree is countable. From this we can infer that many reals with properties that are not normally associated with genericity still seem generic in the context of some perfect tree. For example, there are reals of minimal degree and reals with high information content, such as the theory of second order arithmetic, that are generic relative to some perfect tree.

The proof that the set of reals not n -generic relative to any perfect tree is countable uses ZFC^- and n iterates of the power set of ω . We show that for sufficiently large n , this requirement is sharp and cannot be significantly improved. From this we see that for reasonably high values of n , the set of reals not n -generic relative to any perfect tree is unusually large (rich) for a countable set of this type. It provides a natural example of a set which needs this level of ZFC to be understood.

While the result above holds only for sufficiently high values of n , for lower values the set of reals not n -generic relative to any perfect tree is also rich. By looking at the iterates of the hyperjump, we demonstrate that the set still contains reals of unexpectedly high complexity. Even for $n = 2$, relatively large fragments of arithmetic fail to prove the set is countable. We also begin to characterize the sets that are 1-generic relative to some perfect tree.

These results are in a similar vein to results of Reimann and Slaman [9], who have studied the set of reals which appear random in some context, in this case relative to some continuous measure. Our results for genericity are analogous to what they discovered for randomness in surprisingly many, but not all, instances.

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2 Co-Countably Many Reals

We wish to show that the set of reals not n -generic relative to any perfect tree is countable. D. Martin [7] used Borel determinacy to show that any property which is Borel and cofinal in the Turing degrees is represented on every degree in a cone of Turing degrees. The base of this cone is the complexity of the winning strategy for an associated game.

Theorem 2.1 (Martin [7]). *Let \mathcal{B} be a Borel set of reals such that for every Turing degree \mathbf{d} there is an $\mathbf{e} \geq_T \mathbf{d}$ and an X in \mathbf{e} such that $X \in \mathcal{B}$. Then there is a degree \mathbf{c} such that for all $\mathbf{b} \geq_T \mathbf{c}$ there is a Y in \mathbf{b} such that $Y \in \mathcal{B}$.*

Proof. Consider a two person game where player I constructs a real X and player II constructs a real Y . Play alternates between the players, each adding the next digit to the real they are constructing for their turn. Player I wins iff $Y \leq_T X$ and either $X \not\leq_T Y$ or $X \in \mathcal{B}$.

By Borel determinacy, there exists a winning strategy σ . Suppose σ is a winning strategy for II. Since \mathcal{B} is cofinal in the Turing degrees, let $Z \in \mathcal{B}$ with $\sigma \leq_T Z$. Let I play Z and II play according to σ resulting in Y . But then since $\sigma \leq_T Z$, we have $Y \leq_T Z$ with $Z \in \mathcal{B}$ so I wins for a contradiction. Hence σ must be a winning strategy for I.

Let $Z \geq_T \sigma$ be an arbitrary real in the cone above σ . Have II play Z and I play according to σ , resulting in X . Since $\sigma \leq_T Z$ we have $X \leq_T Z$. Since I wins, $Z \leq_T X$ and $X \in \mathcal{B}$. Hence for any Z in the cone, there is an $X \in \mathcal{B}$ such that $X \equiv_T Z$. \square

Reimann and Slaman [9] have developed a powerful way to relativize this lemma. Let $\mathcal{B} \subseteq 2^\omega \times 2^\omega$ denote a set of reals where the first real holds some property relative to the second. Let $\mathcal{B}^Z = \{X \mid (X, Z) \in \mathcal{B}\}$ and let the notation $X \equiv_{T,A} Y$ mean $X \oplus A \equiv_T Y \oplus A$. Suppose that for every Z the set \mathcal{B}^Z is Borel in Z and cofinal in the Turing degrees as in the above method for generating a cone. They prove that for all but countably many reals X , there exist reals Y and G such that $X \equiv_{T,G} Y$ and $Y \in \mathcal{B}^G$. We outline this proof in the next paragraph.

Let β be the least ordinal such that L_β satisfies enough ZFC (L_β is countable) and let $X \notin L_\beta$ be arbitrary. Reimann and Slaman use Kumabe-Slaman forcing to find a real G such that $L_\beta[G] \models \text{ZFC}$ and every element of $2^\omega \cap L_\beta[G]$ is recursive in $X \oplus G$. In particular, the strategy for the game in the proof of Theorem 2.1 relative to G is recursive in $X \oplus G$. So by Theorem 2.1 relativized to G there exists $Y \in \mathcal{B}^G$ with $Y \equiv_{T,G} X$.

Thus to prove all but countably many reals are n -generic relative to some perfect tree, we need to find a set \mathcal{B} such that for any X, Y, G with $Y \equiv_{T,G} X$ and $Y \in \mathcal{B}^G$ we have X n -generic relative to some perfect tree. \mathcal{B} must also be Borel and such that for every Z the set \mathcal{B}^Z is cofinal in the Turing degrees. We find it suffices to let \mathcal{B} be the set of reals of Turing degree $X \oplus A$ for any X, A such that X is $(n+1)$ -generic(A). We use the following lemma.

Lemma 2.2. *Let $n \geq 2$, A be a set, X be n -generic(A), and $X \equiv_{T,A} Y$. Then Y is $(n-1)$ -generic relative to some perfect tree.*

Proof. Let $\Psi : X \rightarrow Y$ and $\Phi : Y \rightarrow X$ be A -recursive Turing reductions that witness $X \equiv_{T,A} Y$. Since X is at least 2-generic(A), let $p \in X$ be such that $p \Vdash \Phi \circ \Psi = \text{id} \wedge \Psi$ total (i.e. this statement holds for all 2-generic(A) reals extending p). Let $T = \{\sigma \mid \exists q \supseteq p[\sigma \subseteq \Psi(q)]\}$. T is a perfect tree by our choice of p . We claim that Y is $(n-1)$ -generic relative to T .

Let S be an arbitrary $\Sigma_{n-1}^0(T)$ set. We consider the pullback $\Psi^{-1}(S) = \{x \mid \exists y[\Psi(x) \supseteq y \wedge y \in S]\}$. T is $\Sigma_1^0(A)$ so S is $\Sigma_n^0(A)$ and $\Psi^{-1}(S)$ is $\Sigma_n^0(A)$. We will now apply the genericity of X for the pullback to get the genericity of Y for S .

Since X is n -generic(A) we have two possible cases.

Case 1: $\exists n[X|n \in \Psi^{-1}(S)]$. We then let m be such that $Y|m \subseteq \Psi(X|n)$ and $Y|m \in S$.

Case 2: $\exists n \forall q \supseteq X|n[q \notin \Psi^{-1}(S)]$. Let m be such that $\Phi(Y|m) \supseteq X|n$. We will show $Y|m$ witnesses Y is $(n-1)$ -generic relative to T for S (no extension in T of $Y|m$ is in S). Consider an arbitrary $r \in T$ such that $r \supseteq Y|m$. Since $r \in T$, let q be such that $\Psi(q) \supseteq r$ and $q \supseteq p$. We note $q \supseteq \Phi(\Psi(q)) \supseteq \Phi(r) \supseteq \Phi(Y|m) \supseteq X|n$. Hence by the condition for this case, $q \notin \Psi^{-1}(S)$ so $r \notin S$. Since r is arbitrary, for all $r \supseteq Y|m$ with $r \in T$ we have $r \notin S$. \square

We note that a similar proof can be used to show for $n \geq 1$ that sets in the same truth table degree as an n -generic are n -generic relative to some perfect tree.

We can now use the approach outlined above.

Theorem 2.3. *For every $n \in \omega$, the set of reals not n -generic relative to some perfect tree is countable.*

Proof. Fix $n \in \omega$ and let

$$\mathcal{B} = \{(x, z) \mid \exists c \exists h[h \text{ is } (n+1)\text{-generic}(c \oplus z) \text{ and } x \equiv_{T,z} c \oplus h]\}$$

\mathcal{B} is arithmetic (since $c, h \leq_T x \oplus z$) so \mathcal{B} is Borel. Given any reals C and Z , we let H be $(n+1)$ -generic($C \oplus Z$) and $X = H \oplus C$ to get $X \in \mathcal{B}^Z$ with $X \geq_T C$. Hence \mathcal{B}^Z is cofinal in the Turing degrees. By the theorem of Reimann and Slaman [9] noted above, for all but countably many reals Y , there exist X and G such that $Y \equiv_{T,G} X$ and $X \in \mathcal{B}^G$.

Given such X and Y , there exist reals A and H such that $A \oplus H \equiv_{T,G} X$ and H is $(n+1)$ -generic($A \oplus G$). Hence $X \equiv_{T,A \oplus G} H$, so $Y \equiv_{T,A \oplus G} H$. By Lemma 2.2, Y is n -generic relative to some perfect tree. Therefore, all but countably many reals are n -generic relative to some perfect tree. \square

3 ZFC^- and Infinitely Many Iterates of the Power Set of ω Required

If we examine the proof that the set of reals not n -generic relative to any perfect tree is countable, we see that the greatest use of the axioms of ZFC comes from the application of Borel determinacy. The proof uses determinacy of a Π_{n+3}^0 game on ω^ω , so it requires ZFC^- and the existence of n iterates of the power set of ω [6]. We prove that for sufficiently large n this is essentially the best possible result. As a consequence, we show that for any finite k the statement “For all n , the set of reals not n -generic relative to any perfect tree is countable” cannot be proved from ZFC^- and k iterates of the power set of ω . This suggests the set of reals not n -generic relative to any perfect tree is a countable set of considerable size and complexity.

Theorem 3.1. *For every $k \in \omega$ the statement “For all n , the set of reals not n -generic relative to any perfect tree is countable” cannot be proved from $\text{ZFC}^- + “\exists k \text{ iterates of the power set of } \omega”$.*

To prove this theorem we use a template developed by Reimann and Slaman [9] for reals random relative to a continuous measure. We work with the case $k = 0$; the general case follows the same pattern. Let λ be the least ordinal such that $L_\lambda \models \text{ZFC}^-$ and let O be the set of limit ordinals below λ . Let M_α , for $\alpha \in O$, denote master codes. These are the elementary diagrams of canonical countings of L_α . Reimann and Slaman prove the theorem by showing that for some fixed n , for every $\alpha \in O$, the master code M_α is not n -random relative to a continuous measure. Since the set of master codes is not a countable set in L_λ , we then have L_λ does not satisfy “For all n , the set of reals not n -random relative to any continuous measure is countable” but $L_\lambda \models \text{ZFC}^-$. Thus the statement cannot be proved from ZFC^- .

To show this, they assume towards a contradiction that some M_β is n -random relative to the measure μ . It is arithmetic to say that M is a master code for an ω -model of “ $V = L_\alpha$ and α a limit and $\alpha \not\geq \lambda$ ”. Note such an ω -model need not be well-founded. They show it is also arithmetic to require that there exists a fixed $m \in \omega$ such that for all such M and N either one coded model embeds into the other or there is a $\Sigma_m^0(M \oplus N)$ set witnessing the ill-foundedness of one of the coded models.

Reimann and Slaman define a set \mathcal{M} , arithmetic in μ , of such psuedo-master codes which are recursive in μ and not shown to be ill-founded by such a comparison. They then define an order on \mathcal{M} such that the well-founded part of this order, I , is arithmetic in $\mu \oplus M_\beta$ and equals the set of $M \in \mathcal{M}$ which are actual master codes M_α . Since random sets cannot accelerate the calculation of well-foundedness, I is arithmetic in μ .

Let $\gamma \leq \beta$ be least such that $M_\gamma \not\leq_T \mu$. Since $\gamma < \lambda$ there is a real $X \in \text{Def}(L_\gamma) \setminus L_\gamma$. By taking a Skolem hull of the parameters defining X , Reimann and Slaman show that M_γ is arithmetic in I , hence arithmetic in μ , and $M_\gamma \leq_T M_\beta$. Since randomness cannot accelerate arithmetic definability,

$M_\gamma \leq_T \mu$ for a contradiction.

This proof uses only two facts about randomness. Namely, that it cannot accelerate arithmetic definability or calculations of well-foundedness. We can give an abstract summary as follows. Let $R(G, T, n)$ be a Δ_1^1 predicate and suppose that for all numbers l there exists a number n such that for all G and T such that $R(G, T, n)$ and all $k, m < l$ the following two statements hold. First, for any real A , if A is $\Sigma_k^0(T)$ and $\Sigma_m^0(G \oplus T)$ then A is $\Sigma_m^0(T)$. Second, if L is a linear order and WF the well founded part of L such that L is $\Delta_k^0(T)$ and WF is $\Delta_k^0(G \oplus T)$ then WF is $\Delta_k^0(T)$. We can then conclude that for every number k the statement “For all n , the set of reals G such that for no T does $R(G, T, n)$ is countable” cannot be proved from $ZFC^- + \exists k$ iterates of the power set of ω . We note that for some choices of R the statement will not hold at all, in which case this is trivial.

To complete a similar proof for genericity relative to the perfect tree T in place of randomness relative to the measure μ we will demonstrate the corresponding facts for genericity. We will show that for any fixed m and k , and for n sufficiently large relative to m and k , if G is n -generic relative to the perfect tree T , then:

1. If A is $\Sigma_k^0(T)$ and $\Sigma_m^0(G \oplus T)$ then A is $\Sigma_m^0(T)$.
2. If WF is the well founded part of a linear order recursive in T and $WF \leq_T G \oplus T$ then $WF \leq_T T$ (weaker than above, but see below).

To show the first fact, we can routinely relativize to a perfect tree the proof that for reals A, G where G is k -generic and A is Σ_k^0 and $\Sigma_m^0(G)$, we get that A is Σ_m^0 . The second fact suffices for our purposes since if L is $\Delta_k^0(T)$ and G is $(n + k - 1)$ -generic relative to T then G is n -generic(L) relative to T . To prove the second fact, we use the following lemma.

Lemma 3.2. *Let T be a perfect tree and L a linear order of ω where $L \leq_T T$. Let WF be the well founded part of L . Let G be 2-generic relative to T and such that $WF \leq_T G \oplus T$. Then $WF \leq_T T$.*

Proof. By (1.) above, it suffices to show WF is $\Sigma_2^0(T)$. For $c \in \omega$ and D an initial segment of L we use the notation $c \in D$ to mean there is an ordered pair in D which contains c . Since $L \leq_T T$, if T can compute which numbers are in an initial segment then T can compute the initial segment itself.

Let Φ be a T -recursive Turing reduction such that $\Phi(G) = WF$. For $b \in L$ let $P(b)$ be the set of reals which code initial segments of L below b . Let $R(b) \leq_T L$ be the tree defined below such that $P(b)$ is the set of paths through $R(b)$.

$$R(b) = \{\sigma \in 2^{<\omega} \mid \forall m, n < \text{length}(\sigma) [(m \in \sigma \rightarrow m \leq_L b) \wedge ((m \in \sigma \wedge n \leq_L m) \rightarrow n \in \sigma)]\}$$

We define Q as the set of strings in T below which Φ does not split on T .

$$Q = \{\sigma \in T \mid \neg \exists \tau, \gamma \in T [\tau, \gamma \supseteq \sigma \wedge \Phi(\tau) \perp \Phi(\gamma)]\}$$

Suppose for some n , $G|n \in Q$. Then we can calculate whether $m \in WF$ by looking for the first $\sigma \in T$ such that $\sigma \supseteq G|n$ and $[\Phi(\sigma)](m) \downarrow$ and taking its value. Hence $WF \leq_T T$ and we are done. Thus we may assume for all n , $G|n \notin Q$. Since G is 2-generic relative to T , there is an l such that for all $\tau \supseteq G|l$ with $\tau \in T$ we have $\tau \notin Q$.

We will use the fact that $b \in WF$ iff $WF \notin P(b)$. Let $S = \{\sigma \mid \Phi(\sigma) \downarrow \wedge \Phi(\sigma) \notin R(b)\}$. We will determine if $b \in WF$ by checking for the existence of a real which computes an element of $P(b)$ using Φ and is generic for S (some initial segment has no extension in S). Let $\Theta(b)$ be the statement

$$\exists \sigma \in T[\sigma \supseteq G|l \wedge \forall \tau \in T[\tau \supseteq \sigma \rightarrow \Phi(\tau) \in R(b)]]$$

Claim. $b \notin WF \Leftrightarrow \Theta(b)$.

Proof. (\Rightarrow) $WF \in P(b)$ since $b \notin WF$. Hence $\Phi(G) \in P(b)$ so G does not meet S . Since G is 1-generic relative to T , there is a k such that for all $\tau \in T$ with $\tau \supseteq G|k$ we have $\tau \notin S$. Thus $G|k$ witnesses $\Theta(b)$.

(\Leftarrow) Let σ witness $\Theta(b)$. Then for all $\tau \in T$ with $\tau \supseteq \sigma$ we have $\Phi(\tau) \in R(b)$. Also, since $\sigma \supseteq G|l$, for all such τ we have $\tau \notin Q$ so Φ splits on T below τ . Using these facts we can construct a perfect subtree of $R(b)$ by applying Φ to T below σ . Hence $P(b)$ is uncountable. Since there are only countably many well founded initial segments of L , $b \notin WF$. \square

By the claim WF is $\Sigma_2^0(T)$ as desired. Hence $WF \leq_T T$. \square

4 Iterated Hyperjumps

We now look at the set of reals which are n -generic relative to some perfect tree for low values of n . We still find that the set of reals not n -generic relative to any perfect tree is a large countable set. It contains reals of high complexity and its countability cannot be proved in large fragments of second order arithmetic. We show that the finite iterates of the hyperjump, $\mathcal{O}^{(n)}$, are not 2-generic relative to any perfect tree and the iterates $\mathcal{O}^{(\alpha)}$ are not 5-generic relative to any perfect tree for any α below the least λ such that $\sup_{\beta < \lambda}(\beta\text{th admissible}) = \lambda$.

We start with an outline of the proof for the case of \mathcal{O} . This set can be viewed as $\{e \mid U_e \text{ is well-founded}\}$ where U_e denotes the e th recursive tree in $\omega^{<\omega}$. We note \mathcal{O} then has the property that the well-foundedness of subtrees cannot contradict the decision made for the parent tree. This can be characterized by a Σ_2^0 set, S , so that if \mathcal{O} were 2-generic relative to some T then T would be able to calculate \mathcal{O} by tracing subtrees.

Lemma 4.1. \mathcal{O} is not 2-generic relative to any perfect tree.

Proof. Suppose not, witnessed by T . Recall $\mathcal{O} = \{e \mid U_e \text{ is well-founded}\}$. Let h be a recursive function defined by $U_{h(e,\gamma)} = \{\sigma \in U_e \mid \sigma \subseteq \gamma \vee \sigma \supseteq \gamma\}$. Let

$$S = \{\tau \in T \mid \exists n \exists l [\tau(n) = 0 \wedge \neg(\exists \gamma \in U_n)(\exists \theta \in T)[\text{length}(\gamma) \geq l \wedge \theta \supseteq \tau \wedge \theta(h(n, \gamma)) = 0]]\}$$

The set S contains finite strings τ which say some tree U_n is ill-founded, but for some length l , there is no extension of τ in T that says some subtree of U_n with root length at least l is ill-founded. In short, τ says U_n is ill-founded but there is no sequence of extensions in T to witness it.

Let U_n be an arbitrary ill-founded tree ($n \notin \mathcal{O}$) and let Z be an infinite path through U_n . Then the subtrees extending initial segments of Z , $U_{h(n,Z|l)}$ for $l \in \omega$, are also ill-founded. Hence $\mathcal{O}(h(n, Z|l)) = 0$ for every $l \in \omega$ so n does not witness that \mathcal{O} meets S . We conclude that \mathcal{O} does not meet S . Since S is $\Sigma_2^0(T)$ and we have assumed \mathcal{O} is 2-generic relative to T , we let k be such that for any $\sigma \in T$ extending $\mathcal{O}|k$ we have $\sigma \notin S$. We can now use the fact that these extensions are sufficiently well behaved to calculate \mathcal{O} from T .

Claim. For any number e , we have $e \in \mathcal{O} \iff \neg \exists \sigma \in T[\sigma \supseteq \mathcal{O}|k \wedge \sigma(e) = 0]$.

Proof. (\Leftarrow) Let $\sigma = \mathcal{O}|\max(k, e) + 1$. $\sigma \in T$ since \mathcal{O} is a path in T , so $\sigma(e) = \mathcal{O}(e) = 1$. Hence $e \in \mathcal{O}$.

(\Rightarrow) Let $e \in \mathcal{O}$ and suppose the conclusion fails, witnessed by σ . U_e is well-founded since $e \in \mathcal{O}$. We will construct an infinite path through U_e to get the desired contradiction. We use an induction to simultaneously construct paths γ through U_e and θ through T . Let j_0 denote e and j_{m+1} denote $h(j_m, \gamma_{m+1})$. We maintain inductively that $\theta_m(j_m) = 0$.

We begin with $\gamma_0 = \langle \rangle$ and $\theta_0 = \sigma$ and note $\theta_0(j_0) = \sigma(e) = 0$ by our assumption. Let γ_m and θ_m be given. $\theta_m \supseteq \sigma \supseteq \mathcal{O}|k$ so $\theta_m \notin S$. Hence we have

$$\forall n \forall l [\theta_m(n) \neq 0 \vee \exists \alpha \in U_n \exists \beta \in T[\text{length}(\alpha) \geq l \wedge \beta \supseteq \theta_m \wedge \beta(h(n, \alpha)) = 0]]$$

Choosing $n = j_m$ and $l = \text{length}(\gamma_m) + 1$ and noting by our induction hypothesis $\theta_m(j_m) = 0$, we get

$$\exists \alpha \in U_{j_m} \exists \beta \in T[\text{length}(\alpha) \geq \text{length}(\gamma_m) + 1 \wedge \beta \supseteq \theta_m \wedge \beta(h(j_m, \alpha)) = 0]$$

We now let $\gamma_{m+1} = \alpha$ and $\theta_{m+1} = \beta$. We note that $\gamma_{m+1} \supseteq \gamma_m$ since $\gamma_{m+1} \in U_{h(j_{m-1}, \gamma_m)}$ and that $\theta_{m+1}(j_{m+1}) = \theta_{m+1}(h(j_m, \gamma_{m+1})) = 0$, completing the induction. \square

Thus \mathcal{O} is $\Pi_1^0(T)$, contradicting \mathcal{O} being 2-generic relative to T . \square

The next lemma will be used in showing that $\mathcal{O}^{(\alpha+1)}$ is not n -generic relative to any perfect tree, given that $\mathcal{O}^{(\alpha)}$ is not. We prove it by applying the same ideas used in the above lemma to the column of $\mathcal{O}^{(\alpha)}$ which computes \mathcal{O} .

Lemma 4.2. Let $X \geq_T \mathcal{O}$ be 2-generic relative to the perfect tree T . Then $T \geq_T \mathcal{O}$.

Proof. Let Φ be a Turing reduction such that $\Phi(X) = \mathcal{O}$. We define S as before, this time for the image under Φ .

$$S = \{\tau \in T \mid \exists n \exists l [\Phi(\tau)(n) = 0 \wedge \neg(\exists \gamma \in U_n)(\exists \theta \in T)[\text{length}(\gamma) \geq l \wedge \theta \supseteq \tau \wedge [\Phi(\theta)](h(n, \gamma)) = 0]]\}$$

We note S is $\Sigma_2^0(T)$ and X does not meet S . Since X is 2-generic relative to T , we let k be such that for any $\sigma \in T$ extending $X|k$ we have $\sigma \notin S$. We now claim that for any e , we have $e \in \mathcal{O}$ if and only if there does not exist a $\sigma \in T$ with $\sigma \supseteq X|k$ and $[\Phi(\sigma)](e) = 0$. This is proved in substantially the same manner as the claim in the previous lemma. As a result, \mathcal{O} is $\Pi_1^0(T)$. We can now use the fact that generics do not accelerate arithmetic definability. Since X is 2-generic relative to T and $\mathcal{O} \leq_T X$, we get $\mathcal{O} \leq_T T$ as desired. \square

Corollary 4.3. *For all $n \in \omega$, $\mathcal{O}^{(n)}$ is not 2-generic relative to any perfect tree.*

Proof. Fix n and suppose not, witnessed by T . We show by induction on $m \leq n$ that $\mathcal{O}^{(m)} \leq_T T$. Given $\mathcal{O}^{(m)} \leq_T T$, we relativize Lemma 4.2 to $\mathcal{O}^{(m)}$ to get $\mathcal{O}^{(m+1)} \leq_T T$, completing the induction. Hence $\mathcal{O}^{(n)} \leq_T T$, contradicting our assumption that $\mathcal{O}^{(n)}$ is 2-generic relative to T . \square

Corollary 4.4. *The statement “All but countably many reals are 2-generic relative to some perfect tree” fails to hold in Π_1^1 -CA.*

Proof. Consider the standard model of Π_1^1 -CA containing the reals X such that $\exists n[X \leq_T \mathcal{O}^{(n)}]$. The set $\{\mathcal{O}^{(n)} \mid n \in \omega\}$ is not a countable set in this model. \square

To handle limit ordinals, we use a lemma in the style of Enderton and Putnam [2].

Lemma 4.5 (Slaman [12]). *Let A be a set and λ a recursive limit ordinal. Suppose that for all $\beta < \lambda$, $\mathcal{O}^{(\beta)} \leq_T A$. Then $\mathcal{O}^{(\lambda)}$ is $\Sigma_5^0(A)$.*

Proof. We continue to use $\mathcal{O} = \{e \mid U_e \text{ is well founded}\}$ where U_e denotes the e th recursive tree in $\omega^{<\omega}$. Since $\mathcal{O} \leq_T A$ we can define \mathcal{O} from A by noting that U_e is well founded iff U_e has no infinite path recursive in A . Hence \mathcal{O} is uniformly $\Pi_3^0(A)$. Similarly, we can get $\mathcal{O}^\mathcal{O}$ is uniformly $\Pi_4^0(A)$ by $X = \mathcal{O} \oplus \mathcal{O}^\mathcal{O}$ iff

$$(X)_0 = \mathcal{O} \wedge (e \in (X)_1 \leftrightarrow U_e^{(X)_0} \text{ has no infinite path recursive in } A)$$

where $(X)_0$ and $(X)_1$ denote the two columns of X .

We extend this idea to find a uniform definition for $\mathcal{O}^{(\lambda)}$. Fix a system of notations, o , for λ . We have

$$\begin{aligned} (b, k) \in \mathcal{O}^{(\lambda)} &\Leftrightarrow \exists m[\{m\}^A = Y \wedge k \in (Y)_b \wedge [\forall c \forall d[o(c) < o(b) \rightarrow \\ &\quad ((o(c) = o(d) + 1 \rightarrow \Gamma((Y)_d, (Y)_c)) \wedge \\ &\quad (o(c) \text{ a limit ordinal} \rightarrow \forall n \forall p[(Y)_{c_n}(p) = ((Y)_c)_n(p))]]]] \end{aligned}$$

where c_0, c_1, c_2, \dots is the fundamental sequence for $o(c)$ and $\Gamma(X, Z)$ is the statement

$$\forall e[e \in Z \leftrightarrow U_e^X \text{ has no infinite path recursive in } A]$$

Then Γ is $\Pi_4^0(A)$ so $\mathcal{O}^{(\lambda)}$ is $\Sigma_5^0(A)$. \square

If we repeat the proof with $\overline{\mathcal{O}^{(\lambda)}}$ we improve the result slightly to $\mathcal{O}^{(\lambda)}$ is $\Delta_5^0(A)$.

Now we can complete our induction.

Theorem 4.6. *Let λ be the least ordinal such that $\sup_{\beta < \lambda}(\beta \text{th admissible}) = \lambda$. Then for all $\alpha < \lambda$ we have $\mathcal{O}^{(\alpha)}$ is not 5-generic relative to any perfect tree.*

Proof. Suppose not, witnessed by β and T . We define the function f by $f(0) = \omega_1^{CK}$, $f(\delta + 1) =$ least admissible greater than $f(\delta)$, and for limit δ , $f(\delta) = \sup_{\xi < \delta} f(\xi)$. We note that λ is the least fixed point of f . Using the fact that $\omega_1^{\mathcal{O}^{(\delta)}} < \omega_1^{\mathcal{O}^{(\delta+1)}}$ for any δ [10], we see by induction that $f(\delta) \leq \omega_1^{\mathcal{O}^{(\delta)}}$ for all δ .

Let α be least such that $\mathcal{O}^{(\alpha)} \not\leq_T T$. Then $\alpha \leq \beta < \lambda$ so $\alpha < f(\alpha)$. If $\alpha = \xi + 1$ for some ξ then Lemma 4.2 relativized to $\mathcal{O}^{(\xi)}$ would result in a contradiction. Hence α is a nonzero limit ordinal so we can choose $\gamma < \alpha$ such that $\alpha < f(\gamma)$. Then $\alpha < \omega_1^{\mathcal{O}^{(\gamma)}}$ so we can fix a system of notations, o , for α recursive in $\mathcal{O}^{(\gamma+1)}$. Since $\gamma + 1 < \alpha$ we have $\mathcal{O}^{(\gamma+1)} \leq_T T$. We now apply Lemma 4.5 relativized to $\mathcal{O}^{(\gamma+1)}$ for T to get that $\mathcal{O}^{(\alpha)} \leq_T T$ for a contradiction. \square

5 1-generics

In the 1-generic case, we can use a variety of approaches to identify sets of reals that are 1-generic relative to some perfect tree and sets whose members cannot have this property.

A real is said to be ranked if it is a member of a countable Π_1^0 set. Equivalently, a real is ranked if it is a path through a recursive tree with no perfect subtrees. The reader is referred to Cenzer et al. [1] for details on the topic, including a proof that for all recursive ordinals α there is a ranked set of degree $\mathbf{0}^{(\alpha)}$. Here we demonstrate these reals are not 1-generic relative to any perfect tree.

Proposition 5.1. *If X is 1-generic relative to some perfect tree, then X is not ranked.*

Proof. Suppose not. Let X be 1-generic relative to the perfect tree T and a path through the recursive tree U with no perfect subtrees. Let $S = \{\sigma \in T \mid \sigma \notin U\}$. Then S is recursive in T and X does not meet S , so there exists an n such that no $\tau \in T$ extending $X|n$ is in S . Hence for every $\tau \in T$ such that $\tau \supseteq X|n$ we have $\tau \in U$. But then U has a perfect subtree, for a contradiction. \square

It follows from Cenzer et al. [1] and this lemma that there are reals arbitrarily high in the hyperarithmetical degrees which are not 1-generic relative to any perfect tree. We note the proof of Lemma 2.2 can be relativized to start with a perfect tree in place of $2^{<\omega}$. Using this we observe that no Turing degree can contain both a ranked set and a real 2-generic relative to some perfect tree (and no truth table degree a ranked set and a real 1-generic relative to some perfect tree). Hence no Δ_2^0 set is 2-generic relative to some perfect tree, and the degrees $\mathbf{0}^{(\alpha)}$ for any recursive α contain no reals 2-generic relative to some perfect tree.

We can also attempt to classify which reals are 1-generic relative to some perfect tree by use of the r.e. (Ershov) and REA hierarchies. The reader is referred to Jockusch and Shore [4] for details on these hierarchies. We begin by observing that no real whose degree is at a finite level of the REA hierarchy (hence also the r.e. hierarchy) is 1-generic relative to some perfect tree.

Proposition 5.2 (Slaman [11]). *Let $n \in \omega$, X a real of n -REA degree. Then X is not 1-generic relative to any perfect tree.*

Proof. Fix n and X and let W be an n -REA set with $X \equiv_T W$. Let $W_1, W_2, \dots, W_n = W$ witness that W is n -REA; for all $i \leq n$ we have $W_i \leq_T W_{i+1}$ and W_{i+1} is r.e.(W_i). Suppose X is 1-generic relative to T . We show by induction that for all $m \leq n$ we have $W_m \leq_T T$ using the following claim:

Claim. *Let Y be r.e.(T) and $Y \leq_T X$. Then $Y \leq_T T$.*

Proof. It suffices to show \overline{Y} is r.e.(T). Since $Y \leq_T X$, let $\{e\}^X = Y$. Let

$$S = \{q \mid \exists n[\{e\}^q(n) \downarrow = 0 \wedge n \in Y]\}$$

We note $X \notin S$ and S is r.e.(T) since Y is r.e.(T). Hence for some l , every q extending $X|l$ is not in S . We can now describe \overline{Y} by noting that $n \in \overline{Y}$ iff $\exists q \supseteq X|l[\{e\}^q(n) \downarrow = 0]$. Hence \overline{Y} is r.e.(T) as desired. \square

For the induction, given $W_m \leq_T T$ we note that W_{m+1} is r.e.(W_m), hence r.e.(T), and apply the claim to W_{m+1} . As a result $W \leq_T T$ so $X \leq_T T$ for a contradiction. \square

We might next hope to show sets of ω -REA degree are not 1-generic relative to any perfect tree. However, we cannot even do this for sets which are ω -r.e. In proving the Friedberg Inversion Theorem for the truth table degrees, J. Mohrherr [8] showed by a reduction that there is a 1-generic G such that $G \leq_{tt} 0'$, hence G is ω -r.e. Here we provide a direct construction. We use the definition [4] that X is ω -r.e. if for some partial recursive $\psi : \omega \times \omega \rightarrow 2$ we have $X(n) = \psi(b, n)$ where b is least such that $\psi(b, n) \downarrow$.

Proposition 5.3. *There is a 1-generic real which is ω -r.e.*

Proof. In this construction we extend to meet the first r.e. set we find while still looking for earlier r.e. sets skipped over. If we find a set that has been

skipped, we start over again from that point. To limit the number of injuries, we require the n th r.e. set to extend the first n bits of the current string.

We build in stages our generic $X \in 2^\omega$ and the partial recursive witness, ψ , that X is ω -r.e. We also use some numeric variables for bookkeeping. r denotes the r.e. set we are looking at, k_i for $i \in \omega$ the number of corrections at the i -th r.e. set, and m_i for $i \in \omega$ the length of the initial segment of X currently meeting the i -th r.e. set (0 if not yet met). We start with $\psi = \emptyset$, $X = \langle \rangle$, $r = 0$, and $k_i, m_i = 0$ for all i .

At stage $n + 1$ we search simultaneously for i such that $m_i = 0$, $\tau \supseteq X_n \upharpoonright \max(i, \max_{j < i}(m_j))$ with $\text{length}(\tau) > n$, and s to find $\{i\}_s^\tau \downarrow$. If $i > r$ we have found a new r.e. set and add it by letting $m_i = \text{length}(\tau)$, $X_{n+1} = \tau$, $r = i$, and for l such that $\text{length}(X_n) < l \leq \text{length}(X_{n+1})$ we let $\psi(2^l + l - k_l, l) = X_{n+1}(l)$. If instead $i < r$ we have found a r.e. set we have skipped over and restart at that point. We do this by first setting $m_i = \text{length}(\tau)$ and for l such that $\max(i, \max_{j < i}(m_j)) < l \leq m_r$ setting k_l to $k_l + 1$. We next reset X_{n+1} to τ (it will not extend X_n , but will extend $X_n \upharpoonright \max(i, \max_{j < i}(m_j))$). Then, for l such that $\max(i, \max_{j < i}(m_j)) < l \leq m_i$ we extend ψ by letting $\psi(2^l + l - k_l, l) = X_{n+1}(l)$. Finally, for j with $i < j \leq r$ we set $m_j = 0$ and last set $r = i$.

Since we require the n th r.e. set to extend $X \upharpoonright n$, after stage i , the value of $X(i)$ can only be changed when X is changed to meet the n th r.e. set for some $n < i$. By the usual Friedberg-Muchnik counting of injuries, $X(i)$ can be changed at most 2^i times after stage i , so at most $2^i + i$ times in all. Hence $\psi(b, n)$ witnesses X is ω -r.e. by starting with $b = 2^i + i$ and moving b down one every time a correction is made. We see that the resulting X is 1-generic since if X does not meet the i th r.e. set then there is no extension of $X \upharpoonright \max(i, \max_{j < i}(m_j))$ which meets the i th r.e. set. \square

We note that by the REA Completeness Theorem (Jockusch and Shore [4]) this gives that for every $X \geq_T \emptyset^{(\omega)}$ there are sets A and J such that $X \equiv_T A \oplus J$ where J is 1-generic (A).

There is still considerable room left to explore in determining which reals are 1-generic relative to some perfect tree. In particular, it is not yet known if every real not 1-generic relative to any perfect tree is hyperarithmetical.

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